Deriving the Black-Scholes PDE Using a Replicating Portfolio

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1 Set Up

The foundation of the Black-Scholes problem is modeling the stochastic stock process as Geometric Brownian Motion (GBM). Written in SDE form we have:

$$dS(t) = S(t)[\mu dt + \sigma dW(t)] \tag{1}$$

$$S(0) = s \tag{2}$$

Where μ is the mean return on the stock process, σ is the volatility and W(t) is the standard Brownian Motion. One of the crucial assumptions to general Black-Scholes theory is the assumption that both μ and σ are constants. As we will see in the derivation, the "magic" of Black-Scholes allows us to price an option without using the mean return. From here forward we omit the time argument for notational convenience and write S and W rather than $S(t)$ and $W(t)$.

The last piece of information we need to set up the problem is the movement of deterministic processes. Specifically, we define a replicating portfolio. This portfolio will consist of some number (or function) of long shares (which we call Δ) of stock and borrowed capital (denoted B). This replicating portfolio will have the same payoff as the call option at expiration and therefore, by the fundamental theorem of finance, the portfolio value must equal the call option value. We construct this portfolio to be entirely self financing and thus deterministic (non stochastic). In this framework, our deterministic processes satisfy the following differential equation:

$$dB = rBdt \tag{3}$$

Where r is the risk-free interest rate (assumed to be constant in this setting).

2 Creating the Hedging Portfolio and Deriving the BS PDE

In order to price the option, we need to construct a portfolio which will replicate the option exactly. We do this by creating a portfolio which is long Δ shares of stock (where Δ is to be determined) and borrow $B from the bank at interest rate r. With the correct choice of Δ we can make this portfolio deterministic (non stochastic) and self replicating. We note that Δ is not a constant but for notational convenience we omit the time argument (t). We denote the option price as a function C(t,T, S(t), σ, r) and for short hand notation simply denoted as C.

$$C = \Delta S - B \tag{4}$$

The replicating portfolio changes in value by Δ times the stock process minus the change in the amount borrowed. We use the self financing property on the right hand side. This yields the SDE:

$$dC = \Delta dS - dB \tag{5}$$

We apply Ito's formula (with the subscript notation denoting partial derivatives) to the Call option and expand to get the following:

$$dC = C_t dt + C_s dS + \frac{1}{2} C_{ss} (dS)^2$$

$$= C_t dt + C_s S[\mu dt + \sigma dW] + \frac{1}{2} C_{ss} S^2 [\mu dt + \sigma dW]^2$$

$$= C_t dt + C_s S[\mu dt + \sigma dW] + \frac{1}{2} C_{ss} S^2 \sigma^2 dt$$

Now plugging into the equation 5 and using the result from equation 3 for dB:

$$C_t dt + C_s S[\mu dt + \sigma dW] + \frac{1}{2} C_{ss} S^2 \sigma^2 dt = \Delta \cdot dS - rBdt$$

$$= \Delta S[\mu dt + \sigma dW] - rBdt \tag{6}$$

We notice now that the dW terms on either side of the equation must be equal. This yields:

$$C_s S \sigma = \Delta S \sigma$$

And finally solving for Δ we find:

$$C_s = \Delta$$

Now, replacing Δ into both sides of equation (6) and simplifying we get:

$$C_t dt + C_s S[\mu dt + \sigma dW] + \frac{1}{2} C_{ss} S^2 \sigma^2 dt$$

$$= C_s S[\mu dt + \sigma dW] - rBdt \tag{7}$$

$$C_t dt + \frac{1}{2} C_{ss} S^2 \sigma^2 dt + rBdt = 0$$

And we note in the last step that by canceling the $dW(t)$ terms we coincidentally cancel the μ terms which makes the Black-Scholes formulation so useful because we do not need to know μ. We now substitute equation (4) into equation (7), cancel dt from each term and simply to get:

$$C_t + \frac{1}{2} C_{ss} S^2 \sigma^2 + r(\Delta S - C) = 0$$

$$C_t + \frac{1}{2} C_{ss} S^2 \sigma^2 + r\Delta S - rC = 0$$

Substituting in our expression for Δ yields:

$$C_t + \frac{1}{2} C_{ss} S^2 \sigma^2 + rC_s S - rC = 0$$

which is the desired Black-Scholes PDE for a European Call Option:

$$C_t + \frac{1}{2} C_{ss} S^2 \sigma^2 + rSC_s - rC = 0 \tag{8}$$

With terminal condition determined by the option payoff:

$$C(T) = max(S(T) - K, 0); \quad (t < T) \tag{9}$$