Solving the Heat Equation With Green’s Function

Ophir Gottlieb

3/21/2007

1 Setting Up the Problem
The general heat equation with a heat source is written as:

$$u_t(x, t) = k \Delta u(x, t) + g(x, t) \quad \text{in } \Omega \quad (1)$$

$$u(x, 0) = f(x, t) \quad \text{in } \Omega \quad (2)$$

$$u(x, t) = h(x, t) \quad \text{on } \partial \Omega \quad (3)$$

Where $x \leq x_0 \in \Omega$, $0 < t \leq t_0$

Further, we know that $G(x,t)$ satisfies:

$$G_t = -k \Delta G \quad (4)$$

$$G(x, t) = 0 \quad \text{on } \partial \Omega \quad (5)$$

$$G(x, t_0) = \delta(x - x_0) \quad (6)$$

2 Finding the Solution with Green’s Function
We begin by taking equation (1), multiplying by $G(x,t)$ on both sides and integrating on both sides over the volume integral and over time to get:

$$\int_0^{t_0} \int_{\Omega} (G \cdot u_t) dx dt = \int_0^{t_0} \int_{\Omega} (G \cdot k \Delta u) dx dt + \int_0^{t_0} \int_{\Omega} (G \cdot g) dx dt \quad (7)$$

We can re-write the LHS integral as:

$$\int_0^{t_0} \int_{\Omega} (G \cdot u_t) dx dt = \int_{\Omega} \int_{t=0}^{t_0} (G \cdot u) \bigg|_{t=0}^{t} dx - \int_0^{t_0} \int_{\Omega} (G_t \cdot u) dx dt \quad (8)$$

We can re-write the RHS after integrating by parts twice to get:
\[
RHS = \int_0^t \int_\Omega (ku) \Delta G dx dt + \int_0^t \int_{\partial\Omega} kG \frac{\partial u}{\partial n} dS(x) dt - \int_0^t \int_{\partial\Omega} ku \frac{\partial G}{\partial n} dS(x) dt + \int_0^t \int_\Omega (G \cdot g) dx dt
\]  

We notice that we have both terms of equation (4) so we can cancel term 2 on the LHS with term 1 on the RHS. We further know that G=0 on \(\partial\Omega\) which allows us to cancel the second term on the right hand side. Specifically we have:

\[
\int_0^t \int_\Omega (ku) \Delta G dx dt + \int_0^t \int_\Omega (G \cdot u) dx dt = 0 \quad \text{by equation (4)} \tag{10}
\]

\[
\int_0^t \int_{\partial\Omega} kG \frac{\partial u}{\partial n} dS(x) dt = 0 \quad \text{by equation (5)} \tag{11}
\]

Finally we are left with:

\[
\int \int_\Omega (G \cdot u) \bigg|_{t=0}^t dx = - \int_0^t \int_{\partial\Omega} ku \frac{\partial G}{\partial n} dS(x) dt + \int_0^t \int_\Omega (G \cdot g) dx dt \tag{12}
\]

We now write the LHS as:

\[
\int \int_\Omega (G \cdot u) \bigg|_{t=0}^t dx = \int \int_\Omega u \cdot \delta(x-x_0) dx - \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx \tag{13}
\]

Which by the property of the delta function yields:

\[
= u(x_0, t_0) - \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx \tag{14}
\]

Finally, putting it all together (and substituting \(h(x,t)\) for \(u(x,t)\) when we integrate over \(\partial\Omega\)) we find the solution formula to the general heat equation using Green’s function:

\[
u(x_0, t_0) = \int \int_\Omega f \cdot G(x, x_0; 0, t_0) dx - \int_0^t \int_{\partial\Omega} k \cdot h \frac{\partial G}{\partial n} dS(x) dt + \int_0^t \int_\Omega G \cdot g dx dt \tag{15}
\]

This motivates the importance of finding Green’s function for a particular problem, as with it, we have a solution to the PDE. You can see the ”Heat Equation: Solving for Green’s Function” article in order to complete the analysis.