# Heat Equation: Solving for Green's Function

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## 1 Setting Up the Problem

Green's Function for the Heat equation satisfies:

$$G_t = -k\Delta G \tag{1}$$

$$G(x,t) = 0 \ on \ \partial\Omega \tag{2}$$

$$G(x,t_0) = \delta(x - x_0) \tag{3}$$

Where $x \leq x_0 \in \Omega$, $0 < t \leq t_0$. Now we write G(x,t) as:

$$G(x,x_0;t,t_0) = \sum_{n=1}^{\infty} T_n(t)X_n(x) \tag{4}$$

## 2 Finding Green's Function For the Heat Equation

We begin with equation (3) and equation (4):

$$\delta(x - x_0) = G(x,x_0;t_0,t_0) = \sum_{n=1}^{\infty} T_n(t_0)X_n(x) \tag{5}$$

Now we multiply each side by $X_i(x)$ and integrate over the volume integral over $\Omega$. We make use of the fact that the eigen functions are orthogonal. This fact, and the properties of the delta function after integrating yield the terminal condition:

$$X_i(x_0) = T_i(t_0) \tag{6}$$

From the PDE in (1) and the relation in (4) we get:

$$\sum_{n=1}^{\infty} T_n' X_n = -k \sum_{n=1}^{\infty} T_n \Delta X_n \tag{7}$$

We now recall that:

$$-\Delta X_n = \lambda_n X_n \quad where\ \lambda_n\ are\ the\ eigen\ values \tag{8}$$

Plugging this into equation (7) yields:

$$\sum_{n=1}^{\infty} T_n' X_n = k \sum_{n=1}^{\infty} T_n \lambda_n X_n \tag{9}$$

We therefore derive an ODE with terminal condition in equation (6):

$$T_n' = k\lambda_n T_n \tag{10}$$

The ODE is solved, and we find:

$$T_n(t) = X_n(x_0) e^{-k\lambda_n(t_0 - t)} \tag{11}$$

And finally, our solution formula for finding Green's function for the Heat Equation:

$$G(x, x_0; t, t_0) = \sum_{n=1}^{\infty} X_n(x_0) e^{-k\lambda_n(t_0 - t)} \cdot X_n(x) \tag{12}$$