Exploring Arbitrage Opportunities in the Binomial Tree

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1 Set Up

We use the same set up as described in the previous article in which we derived the expressions for the risk neutral probabilities and delta. The single step setup is repeated below for convenience.

We let S_t be the stock price at time t. In this simple example we have only two time periods; $t = 0$ and $t = 1$. We can write the starting stock price as S_0 with only the two possibilities of going up to $u \cdot S_0$ or down to $d \cdot S_0$; where $u > 1.0$ and $d < 1.0$. Whether we move up or down from S_0 we write the resulting price as S_1.

Next we define the interest rate for the single period to be r and assume continuous discounting. Finally we define the *risk-neutral* probabilities of moving up or down as q_u and q_d. The simple set up is illustrated below.

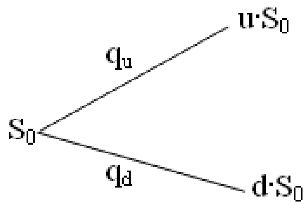

Figure 1: Binomial Tree Setup For Underlying Stock

2 Exploit Arbitrage Opportunities

Claim #1: If $e^{rT} > u$ then an arbitrage can be exploited.

Proof:

$$e^{rT} > u$$

$$\Rightarrow S_0 \cdot e^{rT} > S_0 \cdot u$$

$$\Rightarrow S_0 \cdot e^{rT} > S_0 \cdot d$$

Start: No we create an initial portfolio short 1 share of stock at S_0 and put the proceeds into the bank.

At Maturity (time T): We buy back the stock at price S_T and cover our short position. The money in the bank has grown to $e^{rT} \cdot S_0$.

But we know that S_T can be one of two values:

$$S_T = S_0 \cdot u$$

or

$$S_T = S_0 \cdot d$$

But from above we have already determined that:

$$S_0 \cdot e^{rT} > S_0 \cdot u$$

and

$$S_0 \cdot e^{rT} > S_0 \cdot d$$

So the final portfolio value is ultimately $S_0 \cdot e^{rT} - S_T > 0$ in all states of the world and we have a risk free profit.

Claim #2: If $e^{rT} < d$ then an arbitrage can be exploited.

Proof: We follow the identical methodology but this time we open a portfolio long 1 share of stock at S_0 and borrow $\$S_0$. At maturity we will have a portfolio this time long one share worth S_T and short $e^{rT} \cdot S_0$. Since $S_T - e^{rT} \cdot S_0 > 0$ in all states of the world; we find another risk free profit.